Chaos in an infinite-dimensional dynamical system

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Abstract

Let $H$ be a Hilbert space, $A = A^* \geq m > 0$ be a selfadjoint operator in $H$, possibly unbounded. Consider the following dynamical system:

$$
\dot{u} = -Au + bu, \quad h(t) := b(u(t)) = \frac{(Au, u)}{(u, u)}, \quad u(0) = u_0; \quad \dot{u} = \frac{du}{dt}.
$$

(1)

Here $(u, u)$ is the inner product in $H$, $u_0 \in D(A)$, $D(A)$ is the domain of $A$. Let us formulate our basic results:

1. Problem (1) has a unique solution defined for all $t > 0$ and $\|u(t)\| = \|u_0\| \quad \forall t > 0$.

2. The solution can be found analytically:

$$
u(t) = \frac{e^{-tA}u_0}{(1 - 2 \int_0^t h(s)ds)^{1/2}}, \quad h(t) := (Ae^{-tA}u_0, u_0).
$$

(2)

Without loss of generality one may assume that $\|u_0\| = 1$, and then the solution can be rewritten as:

$$
u(t) = \frac{\int_{-\infty}^{\infty} e^{-\tau t}dE_\tau u_0}{(\int_{-\infty}^{\infty} e^{-\tau t}d\rho(s))^{1/2}}, \quad \rho(s) := (E_\tau u_0, u_0),
$$

(3)

where $E_s = E((-\infty, s])$ is the resolution of the identity of $A$.

3. If the interval $[m, m + \delta]$ does not contain eigenvalues of $A$ and is filled with an absolutely continuous spectrum of of $A$, and if $E([m, m + \delta])u_0 \neq 0$, then the solution to (1) does not have a strong limit in $H$ and does not remain in any fixed finite-dimensional subspace of $H$.

In this sense the solution exhibits chaotic behavior.

4. If $m$ is an isolated eigenvalue of $A$, $P_m$ is the orthoprojector on the corresponding eigenspace $H_m$ of $A$, and if $P_m u_0 \neq 0$, then the limit $\lim_{t \to \infty} u(t) = \phi$ exists, $\phi \in H_m$, and $\|\phi\| = \|u_0\|$.