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Gradient vector fields with impulse action on manifold

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Definition 1 *We say that the four-tuple $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$ is called a vector field with impulse action on M^n , if*

a) X is a smooth vector field on M^n;
b) $\Gamma^{n-1} \subset M^n$ and $\Sigma^{n-1} \subset M^n$ are closed smooth submanifold of codimension 1 (in general unconnected), such that $\Gamma^{n-1} \bigcap \Sigma^{n-1} = \emptyset$;
c) the vector field X is transverse to submanifold Γ^{n-1} Σ^{n-1};
d) $\varphi : \Gamma^{n-1} \to \Sigma^{n-1}$ is a diffeomorphism.

Let $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$ be a vector field with smooth impulse action, $p \in M^n \setminus \Gamma^{n-1}$, and (a, b) be an interval containing 0. Then by **integral curve** we will call a smooth map $\alpha : (a, b) \to M^n$ such that $\alpha(0) = p$, $\alpha(t) \bigcap \Gamma^{n-1} = \emptyset$, and $\alpha^{'}(t) = X(\alpha(t))$. In some case when the integral curve $\alpha : (a, b) \to M^n$ extendts to the value b so that $\alpha(b) \in \Gamma^{n-1}$, then it is calld **disconnected**. This means that the point $\alpha(b)$ is mapped to the point $\varphi(\alpha(b)) \in \Sigma^{n-1}$ and then moves along the integral curve, that passes through the point $\varphi(\alpha(b))$.

Let $f : M^n \longrightarrow [0, 1]$ be a smooth function **with finite number of critical points**. Suppose, that $0 = c_1 < c_2 < ... < c_{k-1} < c_k = 1$ are all critical values of f. Choose regular values p_i, q_i of f such that

$$0 < p_1 < q_1 < c_2 < p_2 < q_2 < c_3 < ... < c_{k-1} < p_{k-1} < q_{k-1} < c_k = 1,$$

and consider submanifolds $M_{p_i} = f^{-1}(p_i)$ and $M_{q_i} = f^{-1}(q_i)$.

Let $\varphi_i(grad_\rho f) : M_{p_i} \longrightarrow M_{q_i}$ and $\varphi_i(grad_\sigma f) : M_{p_i} \longrightarrow M_{q_i}$ be diffeomorpisms constructed using gradient vector fields for Riemannians metrics ρ and σ on M^n. Then we can define the following diffeomorphism:

$$\Phi_i(grad_\sigma f, grad_\rho f) = \varphi_i(grad_\sigma f)^{-1} \cdot \varphi_i(grad_\rho f)$$

which in general is not the identity on M_{p_i}. Denote

$$\Gamma^{n-1} = \cup_i M_{q_i},\ \Sigma^{n-1} = \cup_i M_{p_i},\ \varphi = \cup_i \varphi_i,\ X = grad_\rho f.$$

Definition 2 *By a **disconnected orbit** of i-th floor of a gradient vector field with smooth impulse action $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$ we will call the orbit which starts on submanifold $\dot{\mathcal{D}}_i^n = f^{-1}(q_i, q_{i+1})$ and attain submanifold $M_{q_{i+1}}$.*

Among disconected traectories of i-floor there may exist one such that after the first "meeting" with submanifold $M_{q_{i+1}} \subset \Gamma^{n-1}$ and after application of diffeomorphism φ_{i+1} this moment or after some time move over points of submanifold $\mathcal{E}_{i+1} = f^{-1}[p_{i+1}, q_{i+1})$, that they already "passed". We called such traectories **quasi-closed**.

Theorem 1. *Let M^n be a smooth closed manifold M^n, $f : M^n \longrightarrow [0,1]$ be a smooth function with finite number of critical values, and $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$ be the gradient vector field of f with smooth impulse action constructed using Riemann's metrics ρ σ. If Euler characteristic of a regular hypersurface*

$$\chi(M_{p_{i+1}}) \neq 0,$$

then among disconnected traectories of i-floor always exist quasi-closed trajectory. Intersection of sets of quasi-closed traectories of i-floor with submanifold M_{p_i} is a compact subset in M_{p_i}.

Definition 3 *Let $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$ be a gradient vector field with smooth impulse action on smooth clossed manifold M^n. Suppose, that γ is a quasi-closed trajectory of i-floor. We say that it is orbitally stable if for every ϵ-neigbourhood U_ϵ there exists a δ-neighbourhood $V_\delta \subset U_\epsilon$ that satisfies the following condition: any disconnected trajectory of i-floor γ_1 that starts in V_δ leaves in U_ϵ and then after every "beating" by submanifold $M_{p_{i+1}}$.*

Definition 4 *Let X be a compact space, $F : X \longrightarrow X$ be a homeomorphism, and $y \in X$ be a fixed point of F. We say that point y is quasi-attracting, if for every neighbourhood U of y there exists a smaller neighbourhood $V \subset U$ of this point, such that for every natural number n we have $F^n(V) \subset U$ $(F^n = \underbrace{F \circ F \circ \ldots \circ F}_{n})$.*

Theorem 2. *Let M^n be a smooth clossed manifold, $f : M^n \longrightarrow [0,1]$ be a smooth function with finite number of critical values, and $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$ be a gradient vector field of f with smooth impulse action constructed using Riemann's metrics ρ and σ. Suppose that γ is a quasi-closed trajectory of i-floor of $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$, that intersect the submanifold $M_{p_{i+1}}$ at some point x. Suppose also that x is a fixed quasi-attracting point for the diffeomorphism*

$$\Phi_{i+1}(grad_\sigma f, grad_\rho f) : M_{p_{i+1}} \longrightarrow M_{p_{i+1}}.$$

Than the quasi-closed trajectory γ will be orbitally stable.