Gradient vector fields
with impulse action on manifold

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Definition 1 We say that the four-tuple \((X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)\) is called a vector field with impulse action on \(M^n\), if

a) \(X\) is a smooth vector field on \(M^n\);
b) \(\Gamma^{n-1} \subset M^n\) and \(\Sigma^{n-1} \subset M^n\) are closed smooth submanifolds of codimension 1 (in general unconnected), such that \(\Gamma^{n-1} \cap \Sigma^{n-1} = \emptyset\);
c) the vector field \(X\) is transverse to submanifold \(\Gamma^{n-1} \subset \Sigma^{n-1}\);
d) \(\varphi: \Gamma^{n-1} \to \Sigma^{n-1}\) is a diffeomorphism.

Let \((X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)\) be a vector field with smooth impulse action, \(p \in M^n \setminus \Gamma^{n-1}\), and \((a, b)\) be an interval containing 0. Then by integral curve we will call a smooth map \(\alpha: (a, b) \to M^n\) such that \(\alpha(0) = p\), \(\alpha(t) \cap \Gamma^{n-1} = \emptyset\), and \(\alpha'(t) = X(\alpha(t))\). In some case when the integral curve \(\alpha: (a, b) \to M^n\) extents to the value \(b\) so that \(\alpha(b) \in \Gamma^{n-1}\), then it is called disconnected.

This means that the point \(\alpha(b)\) is mapped to the point \(\varphi(\alpha(b)) \in \Sigma^{n-1}\) and then moves along the integral curve, that passes through the point \(\varphi(\alpha(b))\).

Let \(f: M^n \to [0, 1]\) be a smooth function with finite number of critical points. Suppose, that \(0 = c_1 < c_2 < \cdots < c_{k-1} < c_k = 1\) are all critical values of \(f\). Choose regular values \(p_i, q_i\) of \(f\) such that

\[0 < p_1 < q_1 < c_2 < p_2 < q_2 < c_3 < \cdots < c_{k-1} < p_{k-1} < q_{k-1} < c_k = 1,\]

and consider submanifolds \(M_{p_i} = f^{-1}(p_i)\) and \(M_{q_i} = f^{-1}(q_i)\).

Let \(\varphi_i(\text{grad}_\sigma f): M_{p_i} \to M_{q_i}\) and \(\varphi_i(\text{grad}_\rho f): M_{p_i} \to M_{q_i}\) be diffeomorphisms constructed using gradient vector fields for Riemannians metrics \(\rho\) and \(\sigma\) on \(M^n\). Then we can define the following diffeomorphism:

\[\Phi_i(\text{grad}_\sigma f, \text{grad}_\rho f) = \varphi_i(\text{grad}_\sigma f)^{-1} \cdot \varphi_i(\text{grad}_\rho f)\]

which in general is not the identity on \(M_{p_i}\). Denote

\[\Gamma^{n-1} = \cup_i M_{q_i}, \Sigma^{n-1} = \cup_i M_{p_i}, \varphi = \cup_i \varphi_i, X = \text{grad}_\rho f.\]

Definition 2 By a disconnected orbit of \(i\)-th floor of a gradient vector field with smooth impulse action \((X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)\) we will call the orbit which starts on submanifold \(D^n_i = f^{-1}(q_i, q_{i+1})\) and attain submanifold \(M_{q_{i+1}}\).

Among disconnected trajectories of \(i\)-floor there may exist one such that after the first “meeting” with submanifold \(M_{q_{i+1}} \subset \Gamma^{n-1}\) and after application of diffeomorphism \(\varphi_{i+1}\) this moment or after some time move over points of submanifold \(E_{i+1} = f^{-1}(p_{i+1}, q_{i+1})\), that they already "passed". We called such trajectories quasi-closed.
Theorem 1. Let $M^n$ be a smooth closed manifold $M^n$, $f : M^n \rightarrow [0,1]$ be a smooth function with finite number of critical values, and $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$ be the gradient vector field of $f$ with smooth impulse action constructed using Riemann's metrics $\rho$ and $\sigma$. If Euler characteristic of a regular hypersurface

$$\chi(M_{p+1}) \neq 0,$$

then among disconnected trajectories of $i$-floor always exist quasi-closed trajectory. Intersection of sets of quasi-closed trajectories of $i$-floor with submanifold $M_{p+1}$ is a compact subset in $M_{p+1}$.

Definition 3. Let $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$ be a gradient vector field with smooth impulse action on smooth closed manifold $M^n$. Suppose, that $\gamma$ is a quasi-closed trajectory of $i$-floor. We say that it is orbitally stable if for every $\varepsilon$-neighbourhood $U_\varepsilon$ there exists a $\delta$-neighbourhood $V_\delta \subset U_\varepsilon$ that satisfies the following condition: any disconnected trajectory of $i$-floor $\gamma_1$ that starts in $V_\delta$ leaves in $U_\varepsilon$ and then after every "beating" by submanifold $M_{p+1}$.

Definition 4. Let $X$ be a compact space, $F : X \rightarrow X$ be a homeomorphism, and $y \in X$ be a fixed point of $F$. We say that point $y$ is quasi-attracting, if for every neighbourhood $U$ of $y$ there exists a smaller neighbourhood $V \subset U$ of this point, such that for every natural number $n$ we have $F^n(V) \subset U$ ($F^n = \underbrace{F \circ F \circ \ldots \circ F}_n$).

Theorem 2. Let $M^n$ be a smooth closed manifold, $f : M^n \rightarrow [0,1]$ be a smooth function with finite number of critical values, and $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$ be a gradient vector field of $f$ with smooth impulse action constructed using Riemann's metrics $\rho$ and $\sigma$. Suppose that $\gamma$ is a quasi-closed trajectory of $i$-floor of $(X, \Gamma^{n-1}, \Sigma^{n-1}, \varphi)$, that intersect the submanifold $M_{p+1}$ at some point $x$. Suppose also that $x$ is a fixed quasi-attracting point for the diffeomorphism

$$\Phi_{p+1}(\text{grad}_\sigma f, \text{grad}_\rho f) : M_{p+1} \rightarrow M_{p+1}.$$

Then the quasi-closed trajectory $\gamma$ will be orbitally stable.