Functional wavelet-thresholding-based long-range dependence parameter estimation with missing data

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Abstract. Semiparametric estimators of temporal and spatial long-range dependence parameters are formulated from spectral functional data. The case where such data are affected by additive measurement noise is studied. Specifically, wavelet thresholding techniques are applied to the spectral curves, for removing the noise. Compactly supported wavelets in a neighborhood of zero frequency are considered in the implementation of such techniques. Linear regression is then applied to the log-directional wavelet thresholded transform of the temporal and spatial spectral curves to obtain an approximation of the fractal and strong-dependence parameters. A simulation study is developed to investigate the bias and efficiency of the computational methods proposed for functional parameter estimation of fractal and strong-dependent processes.

Keywords: Fractal spectral processes, long-range dependence parameters, spatiotemporal parametric models, spectral functional data, wavelet transform.

1 Introduction

The statistical analysis of functional data displaying long-range dependence has great interest because of its practical motivation in the analysis of complex biological systems. The functional statistical context provides a suitable framework for the analysis of such systems (see Bosq and Blanke\textsuperscript{2}). In the case where such models display global or local self-similarity, these parameters can be related with fractal parameters (see Kelbert \textit{et al.}\textsuperscript{9}). The classical spectral projection methods, can not be applied in the context of fractal and strong-dependence models, since, in the spectral domain, their functional dependence structures also display heavy tails and high local singularity. Several computational problems then arise in the implementation of functional parameter estimation algorithms that hinder their performance (see Bardet \textit{et al.}\textsuperscript{1}, Frías \textit{et al.}\textsuperscript{3}\textsuperscript{5}\textsuperscript{6}\textsuperscript{7}\textsuperscript{8}, Frías and Ruiz-Medina\textsuperscript{4}, among others).
In this paper, we address the problem of functional parameter estimation of spatiotemporal strong dependence models, from spectral functional data affected by additive spectral measurement noise. The model-based local singularity of such data is processed by their projection in suitable regular compactly supported wavelet function bases. The local variability introduced by the spectral measurement noise is removed by applying wavelet-thresholding techniques. The spatiotemporal semiparametric model family studied, characterized in the spectral domain, is given by the convolution of a separable Riesz kernel (separable fractal kernel) with a second-order process satisfying suitable regularity and weak-dependence conditions (see Frías and Ruiz-Medina[4]). Four wavelet-based functional estimation algorithms are designed for semiparametric estimation of such a family, displaying fractality and strong-dependence. Smoothed wavelet thresholding and spectral techniques are combined for directional estimation of the fractal spectral and strong-dependence parameters. A simulation study is carried out where the performance of the functional parameter estimation algorithms is illustrated, regarding bias, efficiency and discrimination of measurement noise.

2 Preliminaries

In this paper, we consider the strong-dependence spatiotemporal parametric model class defined by the mean square convolution of a separable spatiotemporal Riesz kernel $r$ with a Gaussian random field $Y$, satisfying suitable regularity and moment conditions. That is, the strong-dependence spatiotemporal model considered in this paper is given by

$$X(t, z) = \int_{\mathbb{R}^{d+1}} r(t - s, z - y) Y(s, y) ds dy, \quad (\nu, \beta_1, \ldots, \beta_d) \in (0, 1/2)^{d+1}. $$

where

$$r(t, z) = |t|^{-1+\nu} \prod_{i=1}^{d} |z_i|^{-1+\beta_i}, \quad z \in \mathbb{R}^d. \tag{1}$$

Depending on the local regularity and moment conditions satisfied by the sample-paths of $Y$, random field $X$ can be defined in the strong sense (i.e. pointwise) or in the weak sense (i.e. in terms of test functions, see Ruiz-Medina, Angulo and Anh, 2003). In this paper, we assume that $Y$ satisfies the conditions needed for the pointwise definition of $X$. Note that the Fourier transform $\hat{r}$ of kernel $r$, defined in equation (1) on $\mathbb{R}^{d+1}$, is given by

$$\hat{r}(\omega, \lambda) = |\omega|^{-\nu} \prod_{i=1}^{d} |\lambda_i|^{-\beta_i}. $$

The spectral density $f_X$ of random field $X$ is then given by

$$f_X(\omega, \lambda) = f_Y(\omega, \lambda) |\omega|^{-2\nu} \prod_{i=1}^{d} |\lambda_i|^{-2\beta_i}. \tag{2}$$
In the above spectral density model we can appreciate that the vector parameter \( \theta = (\nu, \beta_1, \ldots, \beta_d) \) determines the local singularity (fractality) of the spectral model, that is, the heavy-tail behaviour of the covariance kernel. The directional self-similarity of the Riesz kernel, given in the spectral domain by function \( g(\omega, \lambda) = |\omega|^{-2\nu} \prod_{i=1}^{d} |\lambda_i|^{-2\beta_i} \), induces a slow decay of function \( f_X \) which is translated into a fractal behavior of our model class in the spatiotemporal domain. In this paper, we will address the problem of semiparametric estimation of the local singularity/fractality of the spectral model (2), when this singularity is increased by the presence of additive spectral noise due to the spectral transforming device measurements.

3 Results and Methodology

In the implementation of the functional estimation algorithms described in this section, we consider the following functional spectral observation model:

\[
Y(\omega, \lambda) = X(\omega, \lambda) + \sigma N(\omega, \lambda),
\]

where \( X \) is a Gaussian process with spectral density given by equation (2). Process \( N \) is a Gaussian spatiotemporal white noise with covariance function defined as

\[
E[N(\omega_1, \lambda_1)N(\omega_2, \lambda_2)] = \delta(\omega_1 - \omega_2)\delta(\lambda_1 - \lambda_2),
\]

with \( \delta \) representing the Dirac Delta distribution in the frequency domain. Parameter \( \sigma \) introduces the local variability due to the observation noise, in the functional spectral sample model.

The spectral density \( f_X \) of our spatiotemporal process \( X \) of interest, defined in equation (2), presents the following asymptotic local fractal behavior, when \( |\omega| \to 0, \) and \( |\lambda_i| \to 0, \) \( i = 1, \ldots, d, \)

\[
f_X(\omega, \lambda) \sim C_1 |\omega|^{-2\nu} \prod_{i=1}^{d} |\lambda_i|^{-2\beta_i}, \quad \nu \in (0, d/2), \ \beta_i \in (0, d/2), \ i = 1, \ldots, d.
\]

In the implementation of the functional estimation algorithms formulated below, compactly supported orthonormal wavelet bases are considered. To remove the local variability \( \sigma \), due to the measurement noise, we applied thresholding rules. Specifically, we eliminate the noise levels in the wavelet transform, or the wavelet coefficients reflecting the local singularity levels due to the noise \( N \). After this step, the multiresolution analysis of a function \( f \) on \( \mathbb{R}^{d+1} \) can be performed in terms of the tensorial product of \( d+1 \) one-dimensional orthonormal wavelet bases. In our case, due to the local asymptotic separability of \( f_X \) in equation (4), we can consider the one-dimensional wavelet transforms of the sample spectral curves available at the temporal and each one of the main spatial directions.
The local asymptotic behavior (4) combined with the wavelet transform properties leads to the following identities for the temporal and spatial directional log-wavelet coefficients of the square root $f^{1/2}_X$ of the spectral density $f_X$ of $X$ (see Frías and Ruiz-Medina [4]):

$$\log_2 f_{j,k}^1 = -j(-\nu + 1/2) + \log_2 C(\psi, \lambda^0),$$

(5)

$$\log_2 f_{j,k}^{1+i} = -j(-\beta_i + 1/2) + \log_2 C(\psi, \omega^0, \ldots, \lambda_{i-1}^0, \lambda_{i+1}^0, \ldots, \lambda_d^0)$$

(6)

for $i = 1, \ldots, d$. The temporal memory parameter $\nu$ and spatial dependence parameters $\beta_i$, $i = 1, \ldots, d$, can then be estimated from the above expressions applying linear regression. The fourth functional estimation algorithms implemented (see Frías and Ruiz-Medina [4]) in the next section are based on identities (5) and (6). Specifically, in Algorithm 1, the log-wavelet coefficients of averaged temporal and spatial spectral curves at different neighborhoods of zero-frequency are considered. In Algorithm 2, the log-wavelet coefficients are computed directly from each temporal and spatial spectral curve at each neighborhood of zero frequency, and after applying linear regression, the functional parameter estimates are averaged. In Algorithm 3, the wavelet transform is implemented as in Algorithm 2, but before applying the logarithmic transform, smoothing is performed over the scale parameter of the wavelet coefficients at each scale. Finally, in Algorithm 4, the wavelet transform is applied to the averaged temporal and spatial spectral curves at each zero-frequency neighborhood, as in Algorithm 1, and also, smoothing is performed over the scale parameter of the wavelet coefficients at each scale. In all the cases, a noise-intensity-dependent threshold is considered, after applying wavelet transform, to eliminate the spectral noise level of the data.

4 Simulations

The implementation of the functional estimation methodologies proposed in the previous section is now illustrated, considering several scenarios within the spatiotemporal model class subsequently described. Spatiotemporal process $X$ is defined as a Gaussian stationary process with spectral density given by

$$f_X(\omega, \lambda_1, \lambda_2) = \begin{cases} 
|\omega|^{-2\nu}|\lambda_1|^{-2\beta_1}|\lambda_2|^{-2\beta_2}, & \text{if } (w, \lambda) \leq (h, k) \\
\exp\left\{-\frac{(w, \lambda - (h, k))^2}{1/2a}\right\} |\omega|^{-2\nu}|\lambda_1|^{-2\beta_1}|\lambda_2|^{-2\beta_2}, & \text{otherwise},
\end{cases}$$

for $a > 0$, and $(h, k)$ sufficiently small. The sample spectral curves are computed from the following spatiotemporal spectral functional data model in a neighborhood $\epsilon(0)$ of the zero-frequency:

$$\hat{X}(\omega, \lambda_1, \lambda_2) = |\omega|^{-\nu}|\lambda_1|^{-\beta_1}|\lambda_2|^{-\beta_2} \varepsilon_1(\omega, \lambda_1, \lambda_2) + \varepsilon_2(\omega, \lambda_1, \lambda_2),$$

(7)
where $\varepsilon_1$ is spatiotemporal spectral white noise generating the process of interest $X$. Process $\varepsilon_2$ is also Gaussian spatiotemporal spectral white noise with intensity $\sigma_{\varepsilon_2}$ independent of $\varepsilon_1$. That is, process $\varepsilon_2$ is identified with process $\sigma N$ in the functional observation model (3). The spectral data model (7) is evaluated in $256 \times 256$ frequency points belonging to the interval $[-127.5 \times 10^{-8}, 127.5 \times 10^{-8}]$, that is, $(\omega, \lambda_1, \lambda_2) \in [-127.5 \times 10^{-8}, 127.5 \times 10^{-8}]$, with discretization step size $10^{-8}$.

From equations (5) and (6), the following estimates are considered:

$$\hat{\nu} = -\hat{\theta}^1 + \frac{1}{2}, \quad \hat{\beta}_i = -\hat{\theta}^{i+1} + \frac{1}{2}, \quad i = 1, 2,$$

where $\hat{\theta}^1$ and $\hat{\theta}^{i+1}, i = 1, 2$, are least squares estimators of the slope in equation (5) and (6), respectively. That is,

$$\hat{\theta}^1 = \frac{\left( \sum_{j=1}^{J} \sum_{k=1}^{K(j)} w_{j,k}^1 - \bar{w}^1 \right) \left( \sum_{j=1}^{J} \sum_{k=1}^{K(j)} g_{j,k} - \bar{g} \right)}{\left( \sum_{j=1}^{J} \sum_{k=1}^{K(j)} g_{j,k} - \bar{g} \right)^2},$$

$$\hat{\theta}^{i+1} = \frac{\left( \sum_{j=1}^{J} \sum_{k=1}^{K(j)} w_{j,k}^{i+1} - \bar{w}^{i+1} \right) \left( \sum_{j=1}^{J} \sum_{k=1}^{K(j)} g_{j,k} - \bar{g} \right)}{\left( \sum_{j=1}^{J} \sum_{k=1}^{K(j)} g_{j,k} - \bar{g} \right)^2}, \quad i = 1, 2,$$

with $\{w_{j,k}^1 = \log_2 \tilde{f}_{j,k}^1, k = 1, \ldots, K(j) \}, \ j = 1, \ldots, J, \ {w_{j,k}^{i+1} = \log_2 \tilde{f}_{j,k}^{i+1}, k = 1, \ldots, K(j), \ j = 1, \ldots, J, \ i = 1, 2 \}$, for $k = 1, \ldots, K(j), \ j = 1, \ldots, J, \ \tilde{f}_{j,k}^1$ and $\tilde{f}_{j,k}^{i+1}, i = 1, 2$, being the discrete wavelet transform coefficients, $J$ being the maximum resolution level considered, $K(j)$ being the number of computed wavelet coefficients at resolution level $j$, $\{g_{j,k} = -j, k = 1, \ldots, K(j), j = 1, \ldots, J \}$, and

$$\bar{w}^1 = \frac{1}{\sum_{j=1}^{J} K(j)} \sum_{j=1}^{J} \sum_{k=1}^{K(j)} w_{j,k}^1, \quad \bar{w}^{i+1} = \frac{1}{\sum_{j=1}^{J} K(j)} \sum_{j=1}^{J} \sum_{k=1}^{K(j)} w_{j,k}^{i+1}, \quad i = 1, 2,$$

$$\bar{g} = \frac{1}{\sum_{j=1}^{J} K(j)} \sum_{j=1}^{J} \sum_{k=1}^{K(j)} g_{j,k}.$$

The simulation study is developed within the following two structural parameter scenarios, corresponding to two extreme cases, slight and heavy spectral singularity, within the range of strong dependence. The noise intensity (values of parameter $\sigma_{\varepsilon_2}$) considered in such cases are also given below.

Case E. $\nu = 0.15, \beta_1 = 0.10, \beta_2 = 0.20 \sigma_{\varepsilon_2} = 0.04$.

Case G. $\nu = 0.35, \beta_1 = 0.4, \beta_2 = 0.3, \sigma_{\varepsilon_2} = 2$. 


In Figures 1 and 2, the estimates of the long-memory and spatial long-range dependence parameters are represented in the two cases above referred, after implementation of the fourth functional estimation algorithms considered, combined with the wavelet thresholded transform. Figures 3 and 4 display the standard deviation estimates of the parameter estimators computed in cases E and G. In the estimates displayed in Figures 1–4, functional estimation algorithms 1, 2, 3 and 4 are implemented from the following spectral curve sample sizes \( n = 16, 36, 100, 400, 900, 2500, 3600, 4900, 6400 \), at temporal and spatial directions. For removing the spectral noise component, case E is implemented considering \( J = 7 \), since this case corresponds to the smoothest model. In case G, wavelet-thresholding methods are applied (wavelet coefficients with absolute value greater than a chosen noise-intensity-dependent threshold are eliminated from the higher resolution level \( j = 8 \)). In this highest local singularity case, discrimination between the structural local variability and the noise local variability is needed. Specifically, in this case, the thresholds considered are \( 0.1436 \times 10^6 \) in the estimation of \( \hat{\nu} \), \( 0.1088 \times 10^6 \) in the estimation of \( \hat{\beta}_1 \) and \( 0.1895 \times 10^6 \) in the estimation of \( \hat{\beta}_2 \).

![Fig. 1](image1.jpg)  ![Fig. 2](image2.jpg)

**Fig. 1.** \( \hat{\nu} \) values (top-left), \( \hat{\beta}_1 \) values, \( \hat{\beta}_2 \) values (top-right), algorithm 1 estimations (blue line), algorithm 2 estimations (red line), algorithm 1 estimations (green line), algorithm 4 estimations (magenta line), parameter value (black line), for case E (\( \nu = 0.15, \beta_1 = 0.10, \beta_2 = 0.20 \)).

**Fig. 2.** \( \hat{\nu} \) values (left), \( \hat{\beta}_1 \) values, \( \hat{\beta}_2 \) values (right), algorithm 1 estimations (blue line), algorithm 2 estimations (red line), algorithm 1 estimations (green line), algorithm 4 estimations (magenta line), parameter value (black line), for case G (\( \nu = 0.35, \beta_1 = 0.4, \beta_2 = 0.3 \)).
5 Final comments

This paper provides a statistical methodology for fractal parameter estimation of spectral functional models associated with complex Biological systems, displaying long-range dependence and fractality. The wavelet-thresholded transform allows to discriminate the noise component, due to spectral transforming device or measurement errors in the spectral domain. Averaging in the spectral domain and/or in the wavelet domain allows to improve efficiency of parameter estimators. The fractal or local singularity level of the functional spectral data, related, in the Gaussian case, with long-range dependence structures, is a feature frequently observed in the statistical analysis of spectral characteristics of functional correlations defining gene expression data and gene microarray data. Finally, we remark that the estimation algorithms implemented here, from functional spectral data affected by additive noise, can also be implemented in the space-time domain, for the analysis of fractal temporal and spatial features in Microbiology and Neurology sciences.

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References


