Basins of Convergence in the Restricted Five-Body Problem of Ollöngren

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Abstract: The paper deals with the chaotic and deterministic character of the basins of convergence in the restricted five-body problem of Ollöngren. The parametric evolution of these regions as well as of the equilibrium locations of the small body is also investigated and some useful remarks concerning this dynamical aspect of the problem are made.
Keywords: Chaotic and deterministic basins of convergence, Newton's method for nonlinear algebraic equations, restricted five-body problem of Ollöngren, numerical determination of equilibrium positions

1. Introduction

The restricted five-body problem of Ollöngren ([4]) describes the dynamics of a small body in the Newtonian field created by four other much bigger bodies, the primaries, three of which have equal masses and are located at the vertices of an imaginary equilateral triangle, while the fourth one with a different mass, is located at the mass center of this configuration. The system is characterized by one parameter, the mass parameter β, which is the ratio of the central mass to a peripheral one. In the relevant international bibliography we can find some other works concerning various aspects of this problem ([3], [5], [6], etc.). In this presentation we deal with the parametric variation of the basins of convergence that are produced by Newton's method when applied to the numerical determination of the equilibrium positions of the small body. These positions are distributed in imaginary circular zones, the number of which depends on the value of parameter β. Each zone is composed by three equilibria. The basins of convergence are formed by the launching points of Newton's method that lead to an equilibrium position of a particular zone. Each basin has a fractal structure and generally consists of a deterministic area which evolves around an equilibrium location and a chaotic region which consists of randomly dispersed points that are mixed with the dispersed points that belong to basins of other equilibrium zones. The nucleus of the deterministic region of each basin is characterized by a very fast convergence (1-5 iterations), while the speed of convergence of the launching points that spread in the chaotic region covers a wide spectrum of values for the number of steps. As the mass parameter β increases, the equilibrium positions approach the imaginary circle of the peripheral primaries, while a transfer of launching points from a basin of a zone to the basin of another zone is observed. As a consequence, the deterministic domains of some basins shrink, while the respective ones of some other basins expand.

2. General characteristics of the dynamical system

2.1. *Symmetries of the configuration*

The configuration of the primaries and the synodic system of coordinates Oxyz are shown in Figure 1. This configuration has three axes of symmetry since it is identified after a rotation of 120^0 around the perpendicular axis Oz. The axes of smmetry are dynamically equivalent.

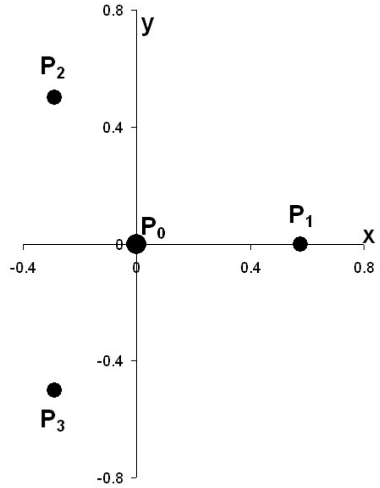


Figure 1. The equilateral triangle configuration of the primaries

2.2. *Equations of planar motion of S*

The motion of the masless body S is described in the synodic coordinate system by the following dimensionless equations

$$\begin{aligned} \ddot{x} - 2\dot{y} &= \frac{\partial U}{\partial x} \\ \ddot{y} + 2\dot{x} &= \frac{\partial U}{\partial y} \end{aligned} \tag{1}$$

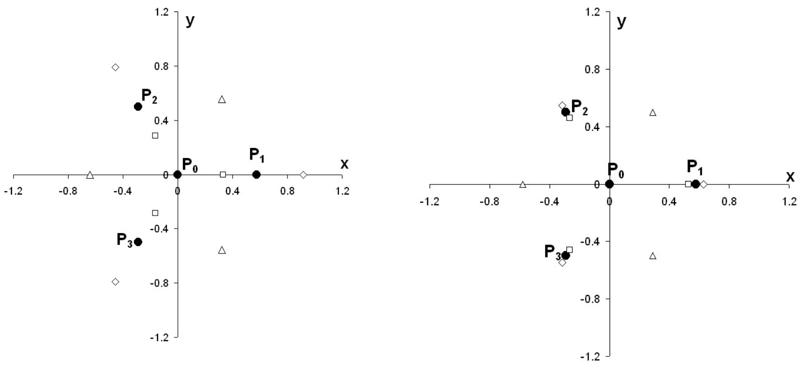
where U is the potential function

$$U(x,y) = \frac{1}{2}\left(x^2 + y^2\right) + \frac{1}{3(1+\sqrt{3}\beta)}\left[\frac{\beta}{r_o} + \sum_{i=1}^{3}\frac{1}{r_i}\right] \tag{2}$$

and r_i, i=0,1,2,3, are the distances of S from the primaries.

2.3. ***Equilibrium locations of S and their parametric variation***

The small body S has fifteen equilibrium positions when β< 0.014. When β>0.014 six of these positions (groups A_2 and B) disappear and the system possesses only nine locations that can be assembled in three groups. These groups (or otherwise equilibrium zones) are symbolized with A_1, C_2 and C_1 by order of appearance from the origin outwards. More specifically, the equilibria of group A_1 are located between the central primary and the peripheral ones. Those of C_1 and C_2 are located outside the triangular configuration of the primaries (Figure 2). The equilibria of each group have the same energy and are characterized by the same state of stability (note that all the equilibria in every group are unstable).



(a) (b)

Figure 2. Distribution of the equilibrium locations in the *xy*-plane in cases: (a) β=2, (b) β=500. The equilibria of A_1 are marked with small rectangulars, the equilibria of group C_2 with small triangles and the equilibria of C_1 with small rhomboids

It is observed that when β increases, the equilibrium points of the three groups approach the peripheral primaries and their energy C decreases. However, their state of stability remains the same. For all values of β>0.014 it holds that C_{A1}>C_{C1}>C_{C2}. However for very large values of β, these differences in C_i, i=A_1, C_1, C_2 are extremely small.

3. Basins of convergence and their parametric variation

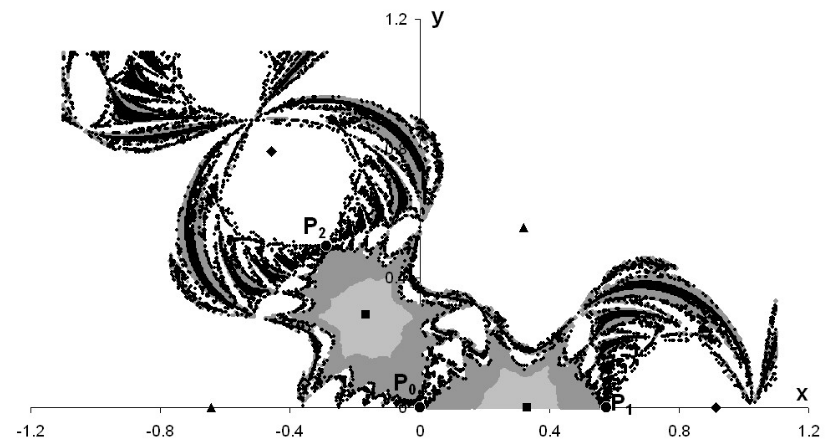
As we have mentioned in the Introduction, the basins of convergence are formed by the launching points of the Newton-Raphson's method when it is applied so that we can find the approximating solutions of the nonlinear algebraic system

$$\begin{aligned} U_x &= 0 \\ U_y &= 0 \end{aligned} \tag{3}$$

when the conditions $\dot{x} = \dot{y} = \ddot{x} = \ddot{y} = 0$ for an equilibrium state hold (see [1] and [2]). We have investigated an orthogonal area of the *xy*-plane that is enclosed between $-1.1<x_0<1.1$ and $0<y_0<1.1$ and we have used a stepsize of the scanning process=0.004. Inside this area are located the primaries and the equilibrium positions. We have also used an accuracy 10^{-8} for the numerical computations. Figures 3, 4 and 5 show these regions for each group of equilibria and for two values of the mass parameter: β=2 and β=500. Table 1 presents some quantitative information for the same values of β, and more specifically about the number of points that belong to each basin for the above values of β. From the exposed material we are able to make some general remarks and we briefly outline the most important among them:

- The basins develop in the *xy*-plane in a way which is consistent with the symmetry of the primaries' configuration.
- In any case, the basin of group C_2 concentrates the majority of the considered launching points of the *xy*-plane, while the one of group C_2 concentrates the minority of the points.
- The basin of convergence of a particular group generally consists of a deterministic nucleus which develops around an equilibrium position that belongs to this group, and of dispersed points that are spread in the neighborhood of this region or are found in a distance from it. These dispersed points are densely mixed with the dispersed points of the basins of other equilibrium groups and create a chaotic domain. This domain is then characterized by high sensitivity to small changes in the values of the launching points.
- The boundaries of the deterministic areas of each basin are not clearly defined since launching points belonging to various basins are densely mixed and this produces a confusing or chaotic picture.
- The deterministic regions often have a fractal structure.
- When β increases, the shape and size of these basins change. The basin of group C_2 enlarges while those of A_1 and C_1 shrink.

The aforementioned basins can also decompose in smaller subgroups, each of which consists of the launching points that converge after a certain number of iterations. This decomposition reveals the parts of the *xy*-plane where the convergence is fast (1-5 iterations), moderate (6-10), or slow (>10 iterations). In Figures 3-5 we have used light gray to show the areas of the first category (fast convergence), gray for the second (moderate convergence) and black for the third one (slow or very slow convergence). The shape and size of these subdomains also change when β increases. Table 1 gives some data concerning this variation in each category. We observe a "transfer" of points from the basins of groups A_1 and C_1 to the basin of C_2 and a simultaneous growth of the latter one.



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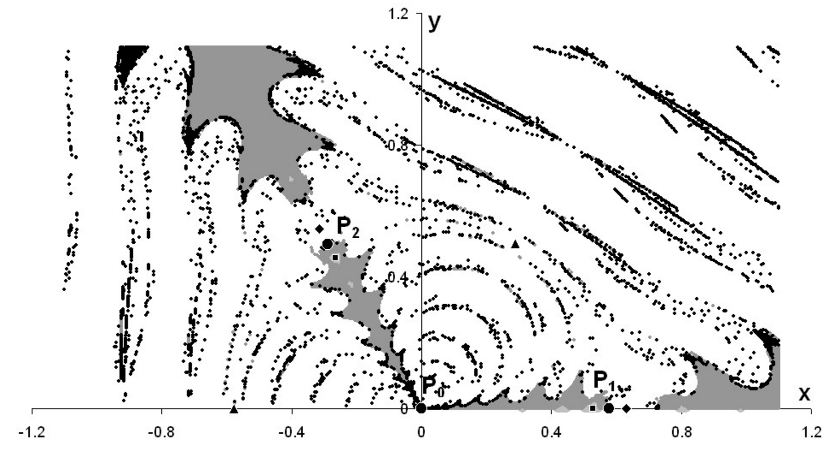
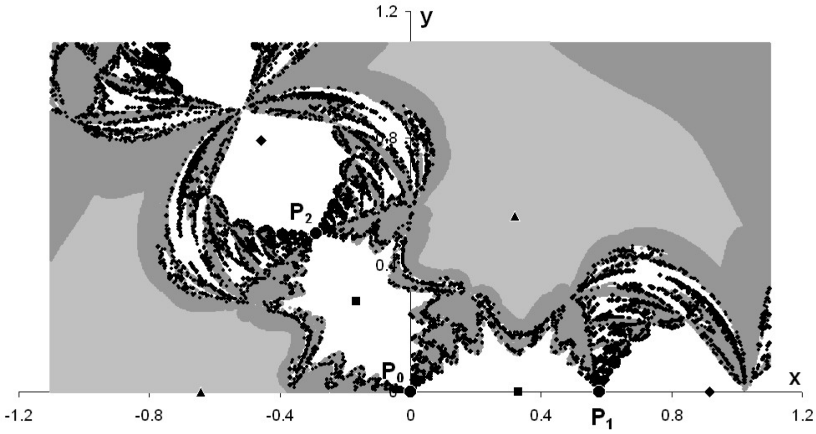


(b)

Figure 3. Basins of convergence of A_1 for classes: 1-5 iterations (light gray), 6-10 iterations (gray), >10 iterations (black).
(a) β=2, (b) β=500



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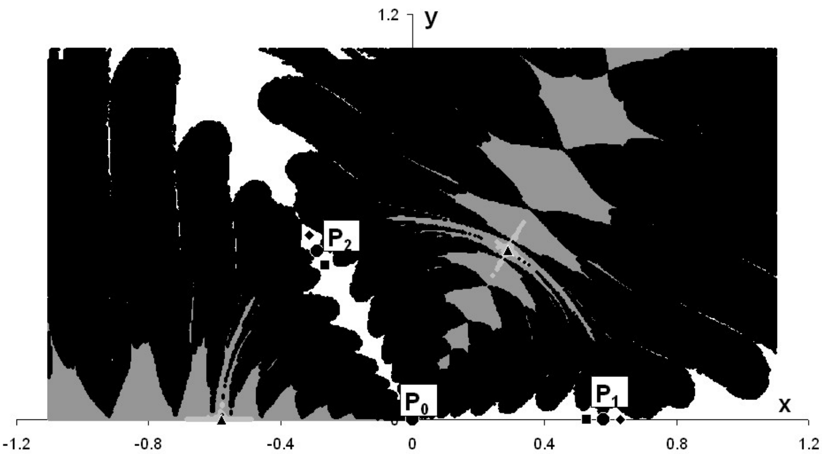
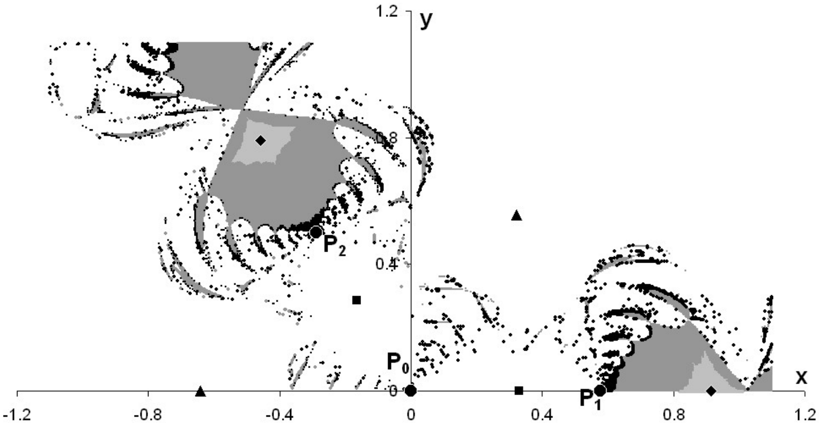


(b)

Figure 4. Basins of convergence of C_2 for classes: 1-5 iterations (light gray), 6-10 iterations (gray), >10 iterations (black).
(a) β=2, (b) β=500



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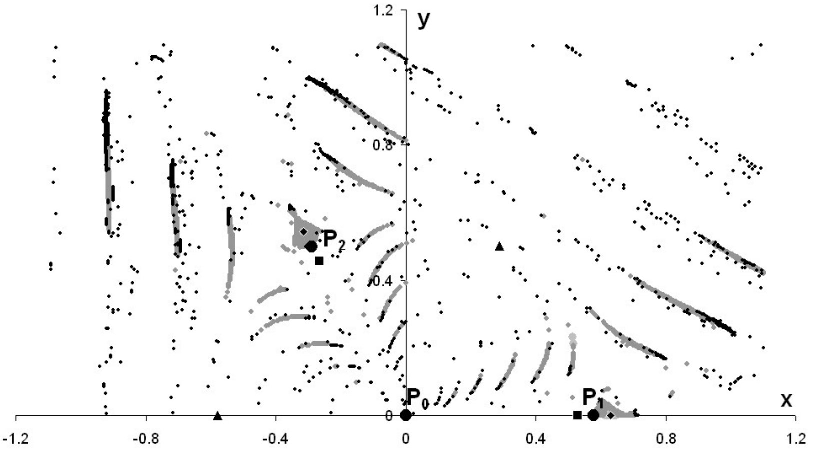

(b)

Figure 5. Basins of convergence of C_1 for classes: 1-5 iterations (light gray), 6-10 iterations (gray), >10 iterations (black).
(a) β=2, (b) β=500

4. Conclusions

In this paper we have examined the chaotic and deterministic aspects of the basins of convergence that are formed by the launching points used in Newton's method for the determination of the equilibrium locations of the small body. For increasing values of the mass parameter the "compact" (or deterministic) areas of A_1 and C_1 that develop around the equilibrium positions of the respective groups, shrink and dissolve, while the dispersed points of these basins lessen. On the contrary, the basin of C_2 grows and expands in the *xy*-plane, occupying more and more of its space.

Equilibrium group (target)	Classes of iterations	Number of launching points	
		β=2	β=500
A_1	1-5	4616	221
	6-10	13690	11679
	>10	8880	4364
	Total points of A_1	27186	16264
C_2	1-5	51801	157
	6-10	46824	21765
	>10	6518	111449
	Total points of C_2	105143	133371
C_1	1-5	1800	97
	6-10	14187	1260
	>10	3191	1058
	Total points of C_1	19178	2415

Table 1. Number of launching points that converge to an equilibrium position of a particular equilibrium group

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