Dynamics of a bouncing ball

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Abstract

The dynamics of a bouncing ball undergoing repeated inelastic impacts with a table oscillating vertically in a sinusoidal fashion is studied using Newtonian mechanics and general relativistic mechanics. An exact mapping describes the bouncing ball dynamics in each theory. We show that, contrary to conventional expectation, the trajectories predicted by Newtonian mechanics and general relativistic mechanics from the same parameters and initial conditions for the ball bouncing at *low speed* in a *weak* gravitational field can rapidly disagree completely. The bouncing ball system could be realized experimentally to test which of the two different predicted trajectories is correct.

1. Introduction

For dynamical systems where gravity does not play a dynamical role, it is conventionally believed [1-4] that if the speed of the system is *low* (i.e., much less than the speed of light *c*), the trajectory predicted by special relativistic mechanics is well approximated by the trajectory predicted by Newtonian mechanics from the same parameters and initial conditions. This conventional belief was however recently [5,6] shown to be false. In particular, it was shown that the two predicted trajectories could rapidly diverge and bear no resemblance to each other.

For dynamical systems where gravity does play a dynamical role but only *weakly*, it is believed that the trajectory predicted by general relativistic mechanics for a *slow-moving* dynamical system is well approximated by the trajectory predicted by Newtonian mechanics from the same parameters and initial conditions. For instance, according to Einstein [1]:

> If we confine the application of the theory [general relativity] to the case where the gravitational fields can be regarded as being weak, and in which all masses move with respect to the co-ordinate system with velocities which are small compared with the velocity of light, we then obtain as a first approximation the Newtonian theory.

Ladipus [7] also wrote:

> For weak fields and low velocities the Newtonian limit is obtained.

We show in this paper with a counterexample dynamical system that this conventional belief is also false.

2. Bouncing ball

Our counterexample dynamical system is a bouncing ball [8,9] undergoing repeated impacts with a table oscillating vertically in a sinusoidal fashion with amplitude A and angular frequency ω. The impact between the ball and the table is instantaneous and inelastic, where the coefficient of restitution α ($0 \le \alpha < 1$) is a measure of the energy lost of the ball at each impact. The table is not affected by the impact because the table's mass is much larger than the ball's mass. In between impacts, the ball moves in a constant gravitational field since the vertical distance travelled is much less than the Earth's radius.

Following [8,9], we will use the ball's velocity v and the table's forcing phase θ at each impact to describe the motion of the bouncing ball. The forcing phase θ is given by $\left(\omega t + \theta_0\right)$ modulus 2π. We will refer to the forcing phase at impact as the impact phase.

In the Newtonian framework, the dynamics of the bouncing ball is [8,9] exactly described by the impact phase map

$$A[\sin(\theta_k)+1] + v_k\left[\frac{1}{\omega}(\theta_{k+1}-\theta_k)\right] - \frac{1}{2}g\left[\frac{1}{\omega}(\theta_{k+1}-\theta_k)\right]^2 - A[\sin(\theta_{k+1})+1] = 0 \qquad (1)$$

and the velocity map

$$v_{k+1} = (1+\alpha)\omega A\cos(\theta_{k+1}) - \alpha\left\{v_k - g\left[\frac{1}{\omega}(\theta_{k+1}-\theta_k)\right]\right\} \qquad (2)$$

where g is the acceleration due to gravity.

In the general relativistic framework, the dynamics of the bouncing ball is (our derivation will be given elsewhere) exactly described by the impact phase map

$$\frac{c^2}{2g}\left\{\frac{1}{2B_k^2}\left\{1+\sin\left[\frac{2B_k g}{c\omega}(\theta_{k+1}-\theta_k)+\Theta_k\right]\right\}-1\right\} - A\left[\sin\left(\theta_{k+1}\right)+1\right] = 0 \qquad (3)$$

where

$$\Theta_k = \sin^{-1}\left(2B_k^2 s_k - 1\right),$$

$$B_k^2 = \frac{s_k - \beta_k^2}{s_k^2},$$

$$s_k = 1 + \frac{2gA\left[\sin\left(\theta_k\right)+1\right]}{c^2},$$

$$\beta_k = \frac{v_k}{c},$$

and the velocity map

$$v_{k+1} = \frac{-c^2\alpha\left(\dfrac{v'_{k+1}-u_{k+1}}{c^2-v'_{k+1}u_{k+1}}\right)+u_{k+1}}{1-\alpha u_{k+1}\left(\dfrac{v'_{k+1}-u_{k+1}}{c^2-v'_{k+1}u_{k+1}}\right)} \tag{4}$$

where

$$u_{k+1} = A\omega\cos(\theta_{k+1})$$

is the table's velocity just after the (k+1)th impact, and

$$v'_{k+1} = \frac{c}{2B_k}\cos\left[\frac{2B_k g}{c\omega}(\theta_{k+1}-\theta_k)+\Theta_k\right]$$

is the ball's velocity just before the (k+1)th impact.

We will show elsewhere that if the ball's velocity and table's velocity are *low* (<< c) and the gravitational field is *weak* ($g << c/\Delta t$, where Δt is the time between impacts), then the general relativistic map [Eqs. (3) and (4)] is approximated by the Newtonian map [Eqs. (1) and (2)].

The impact phase maps Eq. (1) and Eq. (3), which are implicit algebraic equations for θ_{k+1}, must be solved numerically by finding the zero of the function on the left side of the equation given θ_k and v_k. We use the *fzero* function in MATLAB for this purpose. Numerical accuracy of the solutions was carefully checked by varying the tolerances.

3. Results

In the example given here, the parameters of the bouncing ball system are: g = 981 cm/s^2, $c = 2.998\times10^3$ cm/s (we have to use an artificially smaller c value for accurate numerical calculation of the general relativistic map), forcing frequency ($\omega/2\pi$) = 60 Hz, forcing amplitude A = 0.012 cm, and coefficient of restitution $\alpha = 0.5$. The initial conditions are 8.17001 cm/s for the ball's velocity and 0.12001 for the *normalized* impact phase (i.e., impact phase divided by 2π).

The Newtonian and general relativistic trajectories are plotted in Figure 1. The figure shows that the two trajectories are close to each other for a while but they are completely different after 21 impacts although the ball's speed and table's speed remained *low* (about $10^{-3}c$) and the gravitational field is *weak* (g is about $10^{-3}c/\Delta t$). We have checked that the breakdown of agreement between the two trajectories is not due to numerical errors.

4. Conclusion

For a *slow-moving* dynamical system where gravity is *weak*, we have shown, contrary to expectation [1,7], that the trajectories predicted by Newtonian mechanics and general relativistic mechanics from the same parameters and initial conditions could rapidly disagree completely. The bouncing ball system could be realized experimentally [8-10] to test which of the two completely different trajectory predictions is physically correct.

References

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Figure 1

Comparison of the Newtonian and general relativistic trajectories: normalized impact phases (top) and velocities (bottom). Newtonian (general relativistic) values are plotted with triangles (diamonds).