Multifrequency autoresonance and Whitham averaging of integrable systems

Victor Yu.Novokshenov

Institute of Mathematics
Ufa Science Center RAS
(e-mail: novik53@mail.ru)

Abstract. Autoresonance is a powerful technique for controlling the amplitude of nonlinear modes. It is a robust method because, over a broad range of parameters, it does not depend on the details of the system, nor on the amplitude or exact range of the sweeping drive. Autoresonance is usually associated with single frequency mode excitations due to the synchronization and phase lock of various nonlinear modes with the driving force. Despite this we propose a model of multifrequency autoresonance which occur in completly integrable systems. This phenomenon is due to a number of stable invariant tori governed by integrals of motion of the integrable system. The basic autoresonant effect of phase locking appears here as Whitham deformations of the invariant tori. This provides also a possibility to transfer a certain initial n-periodic motion into a given m-periodic motion as a final state.
Keywords: integrable systems, Lax pair, algebro-geometric solutions, theta functions, autoresonance, Whitham deformations, phase locking.

1 Introduction

1. The oscillating frequency of a nonlinear, Duffing-like oscillator changes with amplitude. If you excite such an oscillator by driving it at its linear frequency, the oscillator's amplitude will grow only marginally before its shifting frequency causes it to go out of phase with the drive, after which the oscillator's amplitude will beat back down to zero. By measuring the oscillator's instantenuous frequency and phase, you could use feedback to grow the oscillator's amplitude arbitrarily. But how can you grow the oscillator to high amplitude without feedback?

A general property of weakly driven, nonlinear oscillator is that, under certain conditions, they *automatically* stay in resonance with their drives even if the parameters of the system vary in time and/or space. This phemomenon is called autoresonance. A number of applications in physics and technology is known since 1930s, exploiting autoresonant effect. The most famous are autogenerators of radio frequency [1] and cyclotron acceleration of relativistic particles [2]. In recent time more applications has been found in astronomy and plasma physics [3].

One can easily see a resemblance of the phase locking effect in autoresonant oscillator with adiabatic deformations of completely integrable Hamiltonian systems. The KAM-theory proves this deformation to conserve the motion over Lioville tori of the system for almost all initial data. The Lioville tori, in its turn, are governed by the first integrals, which depend now on a slow variable $\tau = \varepsilon t$. An analytic desciption of the motion is done by the well-known Kuzmak-Whitham [4], [1] method. Here the phase shifts of the quasi-periodic oscillations are strongly matched to the driving frequences, while the first integral's evolution is controlled by an averages of the driving force over basic periods. The latter is done through the solutions of the Whitham equations, which provide elimination of the "secular" terms for the higher-order approximations.

Note that for a Hamiltonian system reduced to canonical action-angle variables $(I, \phi) = (I_1, \ldots, I_n, \phi_1, \ldots, \phi_n)$,

$$\begin{cases} \dot{I} = \varepsilon f(I, \phi, \varepsilon), \\ \dot{\phi} = \phi_0 + \varepsilon g(I, \phi, \varepsilon), \end{cases} \tag{1}$$

this procedure is equivalent to classical multi-phase averaging method, ascending to H.Poincaré [5].

Apply now multi-phase averaging for integrable systems in a reverse way, namely, for some given deformation of n-periodic solution find small driving force, implementing the deformation. More precisely, assume some deformation of the action-angle variables, which transforms initial n-periodic motion to a given m-periodic motion during finite slow time interval $(t \sim O(\varepsilon^{-1}))$. In general, this will cause the drive not to be small, moreover the Hamiltonian structure will fail. To avoid this, restrict the class of deformations to those satisfying Whitham equations. Now the resulting driving force appears to be small and is explicitely controlled by boundary conditions for the Whitham equations. An auxiliary constraints on the angle variables demonstrate specific autoresonant features – the phase locking and synchronization with driving force frequences.

Note that the procedure is essentially multi-phase, which was not known for physically interesting systems. For example, it is possible to drive in adiabatic way a top-like oscillator from a stable state to some given n-periodic rotation.

2 Integration method

The finite-gap integration theory (see, for example [7],[8]) provides a unified approach to the linearization of the phase flow and is based on the Lax form of the equations of motion:

$$\frac{dL(\lambda)}{dt} + [L(\lambda), A(\lambda)] = 0, \tag{2}$$

where L, A, $\Psi = \cdot(\lambda, I, \phi)$ are $l \times l$ matrices, polynomial in λ. Equation (2) provides an unperturbed system (1), e.g., $\dot{I} = 0$, $\dot{\phi} = \phi_0$. As soon as the Lax representation is found, one should construct the Ψ-function, also called the Baker-Akhiezer function, which is a solution of the linear system,

$$L(\lambda)\Psi = \mu\Psi, \tag{3}$$

$$\frac{d\Psi}{dt} = A(\lambda)\Psi. \tag{4}$$

Suppose that the determinant of $L(\lambda) - \mu I$ has the form

$$\det(L(\lambda) - \mu I) = \mu^2 - P_{2g+1}(\lambda) = \mu^2 - (\lambda - \lambda_1)\dots(\lambda - \lambda_{2g+1}). \tag{5}$$

Then equation $\mu^2 = P_{2g+1}(\lambda)$ defines hyperelliptic surface Γ of genus g. Suppose the branch points of Γ are real-valued, $\lambda_1 > \lambda_2 > \dots > \lambda_{2g+1}$, so that the bands $(-\infty, \lambda_{2g+1}], \dots, [\lambda_4, \lambda_3], [\lambda_2, \lambda_1]$ form the spectrum of the operator A in (4).

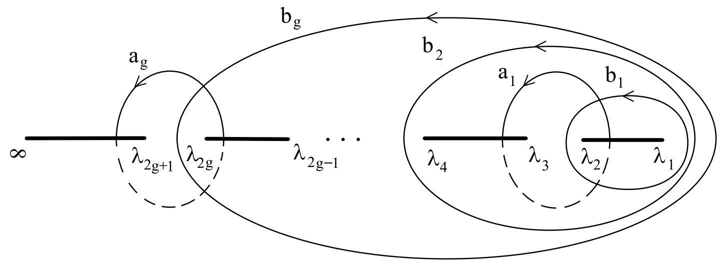

Fig. 1. The spectrum of the operator A and the basis of cycles a, b.

Choose the canonical basis of cycles on Γ as shown at Fig.1 and construct a basis of normalized holomorphic differentials $d\omega_1(p), \dots, d\omega_g(p)$

$$\int_{a_j} d\omega_k(p) = 2\pi i \delta_{jk}, \qquad B_{jk} = \int_{b_j} d\omega_k(p). \tag{6}$$

Here p is a point of Γ "hanging" over λ on the sheets $\mathbf{C} \setminus (\lambda_{2g+1}, \lambda_{2g}) \cup \dots \cup (\lambda_3, \lambda_2)$. The Riemann theta function with characteristics on Γ has standard form

$$\theta[\alpha, \beta]\,(z|B) = \sum_{m \in Z^2} \exp\left[\frac{1}{2}\langle B(m+\alpha), m+\alpha\rangle + \langle z + 2\pi i \beta, m+\alpha\rangle\right], \tag{7}$$

$$B = \{B_{jk}\}, \qquad \theta[0,0]\,(z|B) = \theta(z).$$

Let $d\Omega$ be an Abelian differential of the second kind on Γ with a principal part $d(\sqrt{\lambda})$ at infinity

$$d\Omega = \frac{\lambda^{2g} + a_1\lambda^{2g-1} + \dots + a_{2g}}{2\sqrt{P_{2g+1}(\lambda)}} d\lambda, \tag{8}$$

$$\int_{a_j} d\Omega = 0, \qquad U_j = \int_{b_j} d\Omega, \qquad j = 1, 2, \ldots, g. \tag{9}$$

Theorem 1. *([7]) There exists meromorphic matrix function $\Psi(p,t)$ on $\Gamma \setminus \infty$, satisfying Lax pair equations*

$$L\Psi = \Psi\hat{\mu}, \qquad \Psi_t = A\Psi, \tag{10}$$

where $\hat{\mu} = diag\,(\mu_1, \ldots, \mu_l)$ and $\det(L(\lambda) - \mu_j I) = 0, \quad j = 1, \ldots, l$.

The j-th column of this matrix has the following structure

$$[\Psi(p,t)]_j^T = \Big\{ \ldots, s_m \frac{\theta[\gamma_m](\mathcal{A}(p) + Ut + D + \sigma R)\theta[\delta_m](D + (1-\sigma)R)}{\theta[\gamma_m](\mathcal{A}(p) + D)\theta[\delta_m](Ut + D + R)} \times e^{t\Omega(p) + \kappa_m \omega(p)}, \ldots \Big\}, \tag{11}$$

where $\mathcal{A}(p) = (\int^p d\omega_1, \ldots, \int^p d\omega_g)$ is Abelian map on $Jac(\Gamma)$, p is local parameter on Γ, $\gamma_m, \delta_m = [\alpha_m, \beta_m]$ are theta- characteristics and $D, R \in \mathbf{C}^g$ are constants. Coefficients κ_m, s_m and $\sigma = 0, 1$ are chosen from normalization conditions at $p = 0^{\pm}, \infty^{\pm}$.

Further we restrict Theorem 1 for the hyperelliptic case, i.e., the function Ψ will be defined on the 2-sheet hyperelliptic surface Γ. This is the case for a number of classical dynamical systems, such as Neumann system, special case of Kovalevskaya top [7] and others.

The Lioville torus of the system (2) coinsides with the Jacobian $\mathrm{Jac}(\Gamma)$, the action-angle variables in (1) are ($\varepsilon = 0$)

$$\begin{cases} I = (\lambda_1, \lambda_2, \ldots, \lambda_{2g+1}), \\ \phi(t) = Ut + D. \end{cases} \tag{12}$$

The non-canonical dynamical variables are expressed through the Ψ-matrix. For example, in the n-periodic reduction of the KdV equation, they can be represented by zeros of Ψ lying at the points $p_1 = \big(\gamma_1, \sqrt{P_{2g+1}(\gamma_1)}\big), \ldots, p_g = \big(\gamma_g, \sqrt{P_{2g+1}(\gamma_g)}\big)$. Then the real variables $\gamma_1, \gamma_2, \ldots, \gamma_g$ satisfy the Dubrovin equations [7]

$$\frac{d\gamma_j}{dt} = \frac{2i\sqrt{P_{2g+1}(\gamma_j)}}{\prod\limits_{j \neq k} (\gamma_j - \gamma_k)}, \qquad j = 1, 2, \ldots, g, \tag{13}$$

which are equivalent to the initial system (2).

3 Deformations of Lax equations

Introduce small deformation parameter $\varepsilon \ll 1$ and define "slow time" $\tau = \varepsilon t$. Let the branch points of Γ depend on slow time

$$\lambda_j = \lambda_j(\tau), \quad j = 1, 2 \ldots, 2g + 1,$$

while other components of the Ψ-function are defined exactly as above. The velocity vector $U(\tau)$ needs to be redefined since the t - derivative of $\phi(t) = Ut + D$ is now

$$\frac{d\phi}{dt} = U + \tau U_\tau = U + O(1), \quad \tau > 0.$$

It is clear that the Ψ -function deformed in such a way is no longer close to the initial one.

The way to correct the "secular terms" ascends to Kuzmak-Whitham theory of perturbed dynamical systems [4]. The phase correction term here can be introduced by another Abelian differential $d\Omega_n, \quad n = 1, 2, \ldots,$ such that

a) For $p \to \infty^+$ the following expansion hold

$$\Omega_n(p,\tau) = \int\limits_{p_0}^{p} d\Omega_n = -\lambda^{2n+\frac{1}{2}} + c_1\lambda^{2n-\frac{1}{2}} + \ldots + c_{2n-1}\lambda^{\frac{1}{2}} + O(\lambda^{-\frac{1}{2}}),$$
$$\partial_\tau c_j = 0, \quad j = 1, \ldots, 2n-1, \tag{14}$$

b)

$$\int\limits_{a_m} d\Omega_n = 0, \quad m = 1, 2, \ldots, g. \tag{15}$$

c) If the differential $d\Omega_n$ is real-valued it has exactly g real-valued zeroes and $2n$ zeroes which do not lie on the real axis.

Let V be a vector of b-periods of $d\Omega_n$, i.e. $V = (V_1, \ldots, V_g)$, $V_m = \int\limits_{b_m} d\Omega_n$. Define the deformed Ψ–function as follows

$$[\Psi(p,t,\varepsilon)]_j^T =$$
$$= \Big\{ \ldots, s_m \frac{\theta[\gamma_m](\mathcal{A}(p) + tU(\tau) + \varepsilon^{-1}V(\tau) + D + \sigma R)\theta[\delta_m](D + (1-\sigma)R)}{\theta[\delta_m](tU(\tau) + \varepsilon^{-1}V(\tau) + D + R)\theta[\gamma_m](\mathcal{A}(p) + D)}$$
$$\times \exp\{t\Omega(p,\tau) + \varepsilon^{-1}\Omega_n(p,\tau) + \kappa_m\omega(p)\}, \ldots \Big\}. \tag{16}$$

Here all the components of Ψ -function coincide with those of equation (16 and the Riemann surface $\Gamma = \Gamma(\tau)$ is given by equation (5). Remind that now $\mathcal{A} = \mathcal{A}(p,\tau), \quad B_{ij} = B_{ij}(\tau)$ since the surface Γ depends on τ.

Assume that deformation of $\Gamma = \Gamma(\tau)$ is governed by the following Whitham equation

$$\Big(\tau d\Omega(p,\tau) + d\Omega_n(p,\tau) \Big)\Big|_{\lambda = \lambda_j(\tau)} \partial_\tau \lambda_j(\tau) = 0, \quad j = 1, 2, \ldots, 2g+1. \tag{17}$$

Equations (17) is a self-similar reduction of the well-known quasilinear Whitham system appearing in the modulation theory for evolution equations of KdV type (see [6]). The main reason for the choice of deformation in the form (16) follows from two basic properties of Abelian integrals in (14), (15).

Theorem 2. *If the branch points satisfy the Whitham equations (17), then the following equations hold*

$$\tau\ \partial_\tau d\Omega(p,\tau) + \partial_\tau d\Omega_n(p,\tau) = 0, \tag{18}$$

$$\tau\ \partial_\tau U + \partial_\tau V = 0. \tag{19}$$

Consider now the action of L and A operators over "deformed eigenfunction" $\Psi(p,t,\varepsilon)$ (16). By the above choice of deformation, we expect that equations of the Lax pair will approximate the original ones.

Theorem 3. *[12] The function $\Psi(p,t,\varepsilon)$ (16) is meromorphic and single-valued on Γ/∞ and satisfies the Lax pair equations*

$$L(\lambda)\Psi = \Psi\hat{\mu}, \tag{20}$$

$$\frac{d}{dt}\Psi = (A(\lambda) + \varepsilon H(\lambda))\,\Psi. \tag{21}$$

Here matrices L, A and $\hat{\mu}$ are the same as in (10) for the fixed τ, H is rational matrix in λ.

In conclusion of the section, we can specify more exactly the form of perturbed dynamical system (1), which is the compatibility condition of the Lax pair (20) and (21):

$$\frac{d}{dt}(L\Psi) = L_t\Psi + L(A+\varepsilon H) = (A+\varepsilon H)\Psi\mu = (A+\varepsilon H)L\Psi,$$

so that $L_t + [L,A] = \varepsilon[H,L]$ and

$$\Psi_\tau\Psi^{-1} = \sum_{s=1}^{g+l-1}\frac{H_s}{\lambda-\lambda_s} + H_0, \tag{22}$$

Since the left-hand side here has no poles (22) at the points $\lambda = \lambda_s$, therefore $[H_s, L] = 0, \quad s \neq 0$, which leaves the only matrix H_0 in the right-hand side:

$$L_t + [L,A] = \varepsilon[H_0, L]. \tag{23}$$

4 Structure of the Whitham deformation

1. Monotonic property. Consider now the properties of solutions of the Whitham system (14).

$$(\ \tau d\Omega(p,\tau) + d\Omega_n(p,\tau))\big|_{\lambda\,=\,\lambda_j(\tau)} \quad \partial_\tau\lambda_j(\tau) = 0, \quad j = 1,2,\ldots,2g+1. \tag{24}$$

The equations (24) are exactly a self-similar reduction of the well-known quasilinear Whitham system proposed in [6] for asymptotic integration of KdV equation

$$(\partial_T - S_j(\lambda_1, \ldots, \lambda_{2g+1})\partial_X) \ \ \lambda_j(X,T) = 0, \quad j = 1, 2, \ldots, 2g+1. \qquad (25)$$

where $S_j = -d\Omega_2(\lambda)/d\Omega_1(\lambda)|_{\lambda = \lambda_j}$, Ω_1, Ω_2 are integrals on Γ like (8) and (14), multiplied by t and x respectively in the Ψ - function formula for KdV equation, $T = \varepsilon t, X = \varepsilon x$.

A generalization of the hodograph method was proposed by S.P.Tsarev [9] for an exact integration of the system (25). However, the self-similar system (24) is much simpler one, so it is possible to give a complete description of its solutions in a way discussed in [10] - [12]. Earlier the self-similar solutions of (25) appeared in asymptotics of KdV equations with step-like initial conditions [13]. Here we apply these results in the context of control of the dynamical system driven by g-periodic force in the way described in Section 2. Namely, the following Theorem 4 proved earlier by R.F.Bikbaev and the author in the context of KdV asymptotics, become useful to prove the existence of slow deformation of g-periodic motion with given initial and final states.

Following a notation of [10] - [12] we call a branch point λ_j *a moving point* if $\partial_\tau \lambda_j \neq 0$. It is clear that every equation (24) has two solutions $\tau = -\Omega_n/\Omega$ and $\partial_\tau \lambda_j = 0$. Remind that all branch points are assumed to be real-valued.

A basic fact of monotonicity of the deformation is established by the following

Theorem 4. *[10] [12] There exists $\tau_0 > 0$ such that if the point $\lambda_j(\tau)$ is moving for $0 < \tau < \tau_0$, then*

1) all other branch points are immovable $\partial_\tau \lambda_i(\tau) = 0, \quad i \neq j$.

2) the point λ_j moves from right to left along the real axis, $\quad \partial_\tau \lambda_i(\tau) < 0$,

2. Autoresonant features. The properties of the Whitham deformation discussed above, can prove some principal features of the autoresonance phenomenon, revealed in this construction. We consider the system (23)

$$L_t + [L, A] = \varepsilon[H_0, L], \qquad (26)$$

where the perturbation is defined by (22).

Following [3], first and most important is the possibility to drive the system to arbitrary high amplitudes. To prove this, note that (26) is equivalent to the Dubrovin system (13), where all dynamic variables γ_j oscillate between the neighboring branch points on $\Gamma(\tau)$, i.e., $\lambda_{2j+1}(\tau) \leq \gamma_j \leq \lambda_{2j}$. Thus the amplitudes evolve with the slow evolution of $\lambda_j(\varepsilon t)$. Theorem 4 shows that λ_j can be moved arbitrary far (we need to reverse time in (26), then evolution of λ_j will go from left to right).

In other words, the system (26) is represented in action-angle variables in the form, cf. (12),

$$\begin{cases} I = (\lambda_1(\tau), \lambda_2(\tau), \ldots, \lambda_{2g+1}(\tau)), \\ \phi(t) = U(\tau)t + \varepsilon^{-1}V(\tau) + D(\tau). \end{cases} \tag{27}$$

This yields another autoresonant property: the nonlinearity of phase ("chirp") is small with respect to linear terms. It is due to Whitham equations (18) and (19) in Theorem 2 that

$$\frac{d}{dt}\phi(t) = U(\tau) + \tau U_\tau(\tau) + V_\tau(\tau) + O(\varepsilon) = U(\tau) + O(\varepsilon).$$

Finally, the phase locking property, i.e. the resonance between eigenfrequencies and frequencies of the driving force, is achieved automatically in (26). This is a consequence of "integrability" of perturbation; the deformed system has the Lax pair representation (20) and (21) (see Theorem 3), and its common solution, the Ψ-function (16), comes out as a proper deformation of the original exact Ψ-function (11).

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