Chaos in a Fractional-Order Jerk Model using Tanh Nonlinearity

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Abstract: Chaos in a fractional-order jerk model using hyperbolic tangent nonlinearity is presented. A fractional integrator in the model is approximated by linear transfer function approximation in the frequency domain. Resulting chaotic attractors are demonstrated with the system orders as low as 2.1.

Keywords: Chaos, Fractional order, Jerk model, Hyperbolic tangent nonlinearity.

1. Introduction

Since the discovery of the famous Lorenz chaotic attractor in 1963, chaos has been extensively studied and found ubiquitous applications in scientific, engineering and mathematical communities, for example, in astronomy [1], galaxies [2], quantum chaos [3], circuits and systems [4], chaos-based communications [5-9], medical science [10-12] and chaotic oscillators [13-18]. An integer-order chaotic system, according to the Poincare-Bendixon theorem [19], must have a minimum order of 3 for chaos to appear. For example, the well-known Chua’s circuit is based on three first-order ordinary differential equations (ODEs), whilst Sprott [14] has alternatively proposed chaotic oscillators based on a single third-order ODE called a “jerk” (time derivative of acceleration) function of the form [14-17]:

\[
\frac{d^3x}{dt^3} + A \frac{d^2x}{dt^2} + \frac{dx}{dt} = G(x)
\]  

(1)

or

\[
\frac{da}{dt} + A \frac{dv}{dt} + \frac{dx}{dt} = G(x)
\]  

(2)

where ‘x’, ‘v’ and ‘a’ may represent position, velocity and acceleration of an object, respectively, and G(x) is a nonlinear component such as

\[
G(x) = \begin{cases} 
G_1(x) = |x| - 2 & : A = 0.6 \ [14] \\
G_2(x) = 1.2x - 4.5\text{sgn}(x) & : A = 0.6 \ [14] \\
G_3(x) = -1.2x + 2\text{sgn}(x) & : A = 0.6 \ [14] \\
G_4(x) = -x + 2\tanh(x) & : A = 0.19 \ [20]
\end{cases}
\]

(3)

Recently, fractional-order chaotic systems of orders less than 3 have been demonstrated based on fractional calculus [21,22]. For example, fractional-
order Chua’s circuit of order as low as 2.7 can produce a chaotic attractor [23], whilst chaos in a fractional-order jerk model [24] of orders as low as 2.1 has been demonstrated. The fractional-order jerk model is of the form:

\[
\frac{da}{dt} + A \frac{dv}{dt} + \frac{d^m x}{dt^m} = G(x)
\]  

(4)

where

\[
\frac{d^m x}{dt^m} = v
\]  

(5)

and \(m\) is a fractional number (0 < \(m\) ≤ 1). Equation (5) can be integrated using a fractional integrator in the frequency domain. Although chaos in the fractional-order jerk model (4) has been illustrated using the similar nonlinear components \(G_1(x), G_2(x)\) and \(G_3(x)\) of (3), chaos in the fractional-order jerk model (4) has never been demonstrated using the nonlinear component \(G_4(x) = -x + 2\tanh(x)\) and \(A = 0.19\) of (3).

In this paper, chaos in a fractional-order jerk model using the hyperbolic tangent nonlinearity \(G(x) = -x + 2\tanh(x)\) is presented. The fractional integrator in the model is approximated by linear transfer function approximation in the frequency domain. Resulting chaotic attractors are demonstrated with the system orders as low as 2.1.

2. A Fractional Integrator

In the frequency domain \((s = j\omega)\), a fractional integrator of order ‘\(m\)’ can be represented by a transfer function [22]:

\[
F(s) = \frac{1}{s^m}
\]  

(6)

where the magnitude in the Bode plots is characterized by a slope of -20\(m\) dB/decade. The transfer function (6) may be approximated by pole-zero pairs \((p_i, z_i)\) of the form [22-24],

\[
\frac{1}{s^m} \approx \frac{1}{(1 + \frac{s}{p_T})^N} \frac{\prod_{i=1}^{N-t} (1 + \frac{s}{z_i})}{\prod_{i=0}^{N} (1 + \frac{s}{p_i})} = G_0 \frac{\prod_{i=1}^{N-t} (s + z_i)}{\prod_{i=0}^{N} (s + p_i)}
\]  

(7)

where \(p_T\) is the corner frequency, \(G_0\) is a constant and \(N\) is given by

\[
N = 1 + \text{Integer} \left( \frac{\log \left( \frac{\omega_m}{p_T} \right)}{\log(ab)} \right)
\]  

(8)
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\[ p_0 = p_T 10^{(y/20m)} \]  \hspace{1cm} (9)

\[ a = 10^{(y/10(1-m))} \]  \hspace{1cm} (10)

\[ b = 10^{(y/10m)} \]  \hspace{1cm} (11)

where \( y \) is the discrepancy (dB) between the actual and the approximated lines, and \( \omega_{\text{max}} \) is the frequency bandwidth of the system. Such pole-zero pairs are found as

\[ z_0 = p_0 10^{(y/10(1-m))} \]  \hspace{1cm} (12)

\[ z_r = ap_r(ab)^r \]  \hspace{1cm} (13)

\[ p_r = p_r(ab)^r \]  \hspace{1cm} (14)

Table 1 summarizes a list of integer-order approximation of fractional integrators of order \( m \) using \( y = 2 \) dB, \( \omega_{\text{max}} = 100 \), \( p_T = 0.01 \).

<table>
<thead>
<tr>
<th>( \frac{1}{\omega_{\text{max}}} )</th>
<th>( \omega_{\text{max}} )</th>
<th>( p_T )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( s^3 )</td>
<td>( 1584.8932(s + 0.1668)(s + 27.83) )</td>
<td>( s^3 )</td>
<td>( 79.4328(s + 0.05623)(s + 17.78) )</td>
</tr>
<tr>
<td>( s^3 )</td>
<td>( 39.8107(s + 0.0416)(s + 3.34)(s + 29.94) )</td>
<td>( s^3 )</td>
<td>( 35.4813(s + 0.03831)(s + 1.778)(s + 12.12)(s + 82.54) )</td>
</tr>
<tr>
<td>( s^3 )</td>
<td>( 15.8489(s + 0.03981)(s + 0.2512)(s + 1.585)(s + 63.1) )</td>
<td>( s^3 )</td>
<td>( 10.7978(s + 0.04642)(s + 0.3162)(s + 2.154)(s + 14.68)(s + 100) )</td>
</tr>
<tr>
<td>( s^3 )</td>
<td>( 9.3633(s + 0.06449)(s + 0.578)(s + 5.179)(s + 46.42)(s + 216) )</td>
<td>( s^3 )</td>
<td>( 5.3088(s + 0.1334)(s + 2.371)(s + 4.217)(s + 74.99)(s + 803.1) )</td>
</tr>
<tr>
<td>( s^3 )</td>
<td>( 2.2675(s + 1.292)(s + 215.4) )</td>
<td>( s^3 )</td>
<td>( 2.01292(s + 2.154)(s + 359.4) )</td>
</tr>
</tbody>
</table>
3. Fractional-Order Jerk Model using Tanh Nonlinearity

Figure 1 shows a structural implementation of the fractional-order jerk model described in (4) and (5) using $G(x) = -x + 2\tanh(x)$ and $K = A = 0.19$. For simplicity, let $X = \text{position } 'x'$, $Y = \text{velocity } 'v'$ and $Z = \text{acceleration } 'a'$ of the jerk model. In Fig. 1, the pole-zero block diagram represents the fractional integrator described in (7) and may be approximated by the pole-zero pairs shown in Table 1 where $y = 2 \text{ dB}$.

Figure 1. A fractional-order jerk model using Tanh nonlinearity.

4. Numerical Results

The fractional-order jerk model shown in Fig. 1 is simulated and the order $m$ is reduced from 1 to 0.1. This result in a total system order reduced from 3 to 2.1. For $m = 1$, Figs 2 and 3 show the resulting chaotic attractor in XY and XZ axes, respectively. For $m = 0.9$, Figs 4 and 5 show the trajectories in XY and XZ axes using $G_0 = 3$ and $y = 2 \text{ dB}$. Note that $G_0$ is simply a scaling factor and therefore may not be the same as $G_0$ shown in Table 1. It can be observed that the double-scroll-like attractor is preserved and is not completely destroyed by order reduction. For $m = 0.5$, Figs 6 and 7 display the trajectories in XY and XZ axes using $G_0 = 30$ and $y = 2 \text{ dB}$.

By using Table 1 where the discrepancy $y = 2 \text{ dB}$, the chaotic attractor can be observed for $m = 0.9$ to 0.2. For $m = 0.1$, the chaotic attractors can be observed using $A = 0.02$ and the discrepancy $y = 3 \text{ dB}$, i.e.

$$\frac{1}{s^{m}} = \frac{501.14(s + 0.6811)}{(s + 0.3162)(s + 681.1)}$$

(15)

In (15), $G_0 = 501.14$. Figures 8 and 9 show the trajectories in XY and XZ axes using $m = 0.1$, $G_0 = 1500$. Figures 10 and 11 show the trajectories in XY and XZ axes using $m = 0.1$, $G_0 = 2000$. Figures 12 and 13 show the trajectories in XY and XZ axes using $m = 0.1$, $G_0 = 3000$. 

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Figure 2. Trajectory in XY axes for $m = 1$.

Figure 3. Trajectories in XZ axes for $m = 1$.

Figure 4. Trajectory in XY axes for $m = 0.9$, $G_o = 3$, $y = 2$ dB.

Figure 5. Trajectories in XZ axes for $m = 0.9$, $G_o = 3$, $y = 2$ dB.

Figure 6. Trajectory in XY axes for $m = 0.5$, $G_o = 30$, $y = 2$ dB.

Figure 7. Trajectories in XZ axes for $m = 0.5$, $G_o = 30$, $y = 2$ dB.

Figure 8. Trajectory in XY axes, for $m = 0.1$, $G_o = 1500$, $y = 3$ dB, $A = 0.02$.

Figure 9. Trajectories in XZ axes, for $m = 0.1$, $G_o = 1500$, $y = 3$ dB, $A = 0.02$. 
5. Conclusions
In this paper, chaos in a fractional-order jerk model using the hyperbolic tangent nonlinearity has been presented. A fractional integrator in the model has been approximated by linear transfer function approximation in the frequency domain. Resulting chaotic attractors have been observed with the system orders as low as 2.1.

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References
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